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A NONHOMOGENEOUS INTEGRODIFFERENTIAL EQUATION IN HILBERT SPACE

Ralph W. Carr[1] and Kenneth B. Hannsgen[2]

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ABSTRACT

Let $y(t,x,f)$ denote the solution of $y'(t) + \int_0^t [d + a(t-s)]L\, y(s)ds$ $f(t)$, $t \geq 0$, $y(0) = x$, where $d \geq 0$ and L is a self-adjoint densely defined operator on a Hilbert space $\mathcal{H}$ with $L \geq \Lambda > 0$. Let $U(t)x = y(t,x,0)$. By analyzing a related scalar equation with parameter, we find sufficient conditions on the kernel $a(t)$ for $\|U(t)\| \to 0$ $(t \to \infty)$ and $\int_0^\infty \|U(t)\| dt < \infty$. These results and a resolvent formula can be combined to reveal the behavior of $y(t,x,f)$ as $t \to \infty$.

AMS(MOS) Subject Classification - 45J05, 45M05, 45M10, 45N05

Key Words - Asymptotic, Convex, Hilbert Space, Integrodifferential Equations, Self-adjoint Linear Operator, Volterra Equations.

Work Unit Number 1 - Applied Analysis

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EXPLANATION

This report provides sufficient conditions on the kernel of a certain class of abstract linear integrodifferential equations in Hilbert space which can be used to study the asymptotic nature of the solution as $t \to \infty$.

As a model problem consider the linear partial integrodifferential equation

$$y_t(t,x) - \int_0^t [d + a(t-s)]y_{xx}(s,x)ds = f(t,x)$$

$$y(t,0) = y(t,\pi) = 0$$

$$y(0,x) = y_0(x) \qquad (0 \le x \le \pi)$$

where $d \geq 0$ is a fixed constant while a and f are known functions satisfying certain prescribed conditions. Our results and a resolvent formula can be combined to reveal the behavior of $y(t,x)$ as $t \to \infty$.

In this problem, which can be regarded as a vibrating string with memory, it is critical that the operator $L = -\frac{\partial^2}{\partial x^2}$ is positive and self-adjoint, i.e.

$$\int_0^\pi Lu(x) \cdot u(x)dx \geq \Lambda \int_0^\pi u^2(x)dx > 0$$

and

$$\int_0^\pi Lu(x) \cdot v(x)dx = \int_0^\pi u(x) \cdot Lv(x)dx$$

for some fixed $\Lambda > 0$ and for every choice of u and v which are twice continuou[illegible] differentiable functions satisfying the boundary conditions $u(0) = u(\pi) = v(0) = v(\pi) = 0$. The result generalizes to include a large class of positive, self-adjoint operators.

A NONHOMOGENEOUS INTEGRODIFFERENTIAL
EQUATIONS IN HILBERT SPACE

Ralph W. Carr[1] and Kenneth B. Hannsgen[2]

1. Introduction. Let $\underset{\sim}{L}$ be a self-adjoint (possibly unbounded) linear operator on a Hilbert space $\mathcal{H}$, with spectral decomposition

$$\underset{\sim}{L}\underset{\sim}{x} = \int_{-\infty}^{\infty} \lambda \, d\underset{\sim}{E}_\lambda \underset{\sim}{x}$$

for $\underset{\sim}{x}$ in $\mathcal{D}$, the domain of $\underset{\sim}{L}$. We assume that the spectrum of $\underset{\sim}{L}$ is contained in an interval $[\Lambda,\infty)$ with $\Lambda > 0$, so that $\underset{\sim}{L}$ is a positive operator. We study the initial value problem

(1.1) $$\underset{\sim}{y}'(t) + \int_0^t [d + a(t-s)] \underset{\sim}{L} \underset{\sim}{y}(s)ds = \underset{\sim}{f}(t) \qquad (t \geq 0) ,$$

(1.2) $$\underset{\sim}{y}(0) = \underset{\sim}{y}_0$$

($' = d/dt$), where $\underset{\sim}{y}_0$ and $\underset{\sim}{f}(t)$ belong to $\mathcal{H}$, $d \geq 0$, and the real-valued kernel a satisfies

(1.3) $a \in C(R^+) \cap L^1(0,1)$. a is nonnegative, nonincreasing, and convex on R^+, $0 < a(0+) \leq \infty$, and $a(\infty) = 0$.

(In this paper, $R^+ = (0,\infty)$, $\overline{R}^+ = [0,\infty)$.) See [9] for a discussion (with references and an example) of applications of (1.1) to viscoelasticity theory.

The resolvent kernel of (1.1) is defined by the formula

(1.4) $$\underset{\sim}{U}(t) = \int_{-\infty}^{\infty} u(t,\lambda) d\underset{\sim}{E}_\lambda ,$$

where $u(t,\lambda)$ is the solution of the scalar problem

(1.5) $$u'(t) + \lambda \int_0^t [d + a(t-s)]u(s)ds = 0, \quad u(0) = 1$$

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with parameter λ $(\Lambda \leq \lambda < \infty,\ 0 \leq t < \infty)$. Under certain additional conditions on $a(t)$, we shall show in Theorem 2.2 that

(1.6)
$$\sup_{\Lambda \leq \lambda < \infty} |u(t,\lambda)| \to 0 \quad (t \to \infty) \quad ,$$

(1.7)
$$\int_0^\infty \sup_{\Lambda \leq \lambda < \infty} |u(t,\lambda)| dt < \infty \quad .$$

It is clear that (1.6) and (1.7) imply, respectively,

(1.8)
$$\|\underset{\sim}{U}(t)\| \to 0 \quad (t \to \infty) \quad \text{and} \int_0^\infty \|\underset{\sim}{U}(t)\| dt < \infty \quad .$$

In view of the resolvent formula

(1.9)
$$\underset{\sim}{y}(t) = \underset{\sim}{U}(t)\underset{\sim}{y}_0 + \int_0^t \underset{\sim}{U}(t-s)\, \underset{\sim}{f}(s) ds$$

for the solution of (1.1), (1.2) (see Theorem 2.1), (1.8) shows, for instance, that $\underset{\sim}{y}(t)$ has a limit in $\mathcal{V}$ $(t \to \infty)$ if $\underset{\sim}{f}(t)$ does.

Our results extend those of [10] with respect to the conditions on $a(t)$ as $t \to 0$, $t \to \infty$. In particular, our results imply that (1.6) and (1.7) hold if a satisfies (1.3) and $-a'$ is convex.

This paper is based, in part, on the first author's Ph.D. thesis, being written at the University of Wisconsin under the supervision of Professor John A. Nohel. His help in the preparation of this paper is gratefully acknowledged.

2. *Statement of results*. A solution of (1.1) is a continuously differentiable function $\mathbf{y}$ from $\overline{R}^+$ to $\mathcal{V}$ such that $\mathbf{L}\,\mathbf{y}(t)$ is continuous in t on $\overline{R}^+$ and (1.1) holds. Hille and Phillips [11, pp. 58-89] give the general theory of Bochner integration, which we shall use in studying (1.1) and (1.9). See [19] for the functional calculus of self-adjoint operators.

Our first result, to be proved in Section 3, summarizes some earlier work and establishes the resolvent formula.

THEOREM 2.1. (i) *Let* (1.3) *hold*. *Then the operator* $\mathbf{U}(t)$ *defined by* (1.4) *and* (1.5) *is bounded on* $\mathcal{V}$ *with* $\|\mathbf{U}(t)\| \leq 1$ $(t \in \overline{R}^+)$. $\mathbf{U}(t)$ *commutes with* $\mathbf{L}$ *on* $\mathcal{D}$ *and is strongly continuous on* $\overline{R}^+$.

(ii) *If* $\mathbf{y}_0 \in \mathcal{D}$, *if* $\mathbf{f}\colon \overline{R}^+ \to \mathcal{V}$ *is continuous with* $\mathbf{f}(t) \in \mathcal{D}$ *for all* t, *and if* $\mathbf{L}\,\mathbf{f}$ *is Bochner integrable on each finite subinterval of* R^+, *then* (1.9) *gives the unique solution of* (1.1), (1.2).

Remark. If $\mathbf{y}_0$, $\mathbf{f}$ are in $\mathcal{V}$ but not necessarily in $\mathcal{D}$, then (as shown in [8] for constant $\mathbf{f}$) (1.9) gives the unique weak solution of an integrated form of (1.1), (1.2).

In proving (1.7), we shall need the technical hypothesis

(2.1) $a(t) = b(t) + c(t)$, *where* b *and* c *satisfy* (1.3) *except that either* $b(0+) = 0$ *or* $c(0+) = 0$ *is permitted*. *Moreover*,

$$\text{(i)} \quad \int_1^\infty t^{-1}\, b(t)\,dt < \infty \quad \textit{and}$$

$$\text{(ii)} \quad -c' \ \textit{is convex on}\ R^+ .$$

The Fourier transform of a will be denoted

$$\hat{a}(\tau) \equiv \varphi(\tau) - i\tau\theta(\tau) \equiv \int_0^\infty e^{-i\tau t}\, a(t)\,dt \ . \tag{2.2}$$

Under hypothesis (1.3), $\hat{a}(\tau)$ is continuous, and $\varphi(\tau)$ and $\theta(\tau)$ are nonnegative for $\tau > 0$ [4].

The frequency conditions

$$(2.3)\qquad \begin{array}{ll} \text{(i)} & \varphi(\tau) > 0 \qquad (\tau > 0) \\ \text{(ii)} & \limsup_{\tau\to\infty} \dfrac{\theta(\tau)}{\varphi(\tau)} < \infty \end{array}$$

are crucial for (1.6) and (1.7); we indicate briefly their role. From [4] we know that if (1.3) holds, (2.3i) fails to hold if and only if $a(t)$ is piecewise linear with changes of slope only at integer multiples of a single positive number; $u(t,\lambda)$ is then asymptotic $(t\to\infty)$ to a nonconstant periodic function, so neither (1.6) nor (1.7) holds. If, on the other hand, (1.3) and (2.3i) hold, a result of Shea and Wainger [20] shows that

$$\int_0^\infty |u(t,\lambda)|dt < \infty \qquad (\lambda > 0) \quad .$$

It is then easy to show from (1.5) that

$$(2.4)\qquad \hat{u}(\tau,\lambda) = \frac{1}{\lambda[\varphi(\tau) + i\tau(\lambda^{-1} - \theta(\tau) - d\tau^{-2})]} \qquad (\tau > 0) \quad .$$

In proving Theorem 2.2, we shall show that if λ is sufficiently large, $\frac{d}{\tau^2} + \theta(\tau) = \lambda^{-1}$ for exactly one positive number $\tau = \omega(\lambda)$ with $\omega(\lambda)$ continuous and $\omega(\lambda)\uparrow\infty$ as $\lambda\uparrow\infty$. From (2.4), it follows that

$$\int_0^\infty |u(t,\lambda)|dt \geq |\hat{u}(\omega(\lambda))| \geq \theta(\omega(\lambda))/\varphi(\omega(\lambda)) \quad .$$

This shows the necessity of (2.3ii) for (1.7).

THEOREM 2.2. *Assume that* (1.3) *holds. Then*

(i) (1.6) *holds if* (2.3) *holds,*

(ii) (1.7) *holds if* (2.1) *and* (2.3) *hold, and*

(iii) *if* (1.7) *holds, then* (2.3) *holds.*

If (1.3) and (2.2i) hold and $a(0+) < \infty$, then (2.2ii) holds if and only if $a(t)$ is strongly positive (that is, $(1 + \tau^2)\varphi(\tau)$ is bounded away from zero; see [10]). In Section 7 below we shall given an example (with $a(0+) = \infty$) where $a(t)$ is strongly positive but (2.2ii) does not hold. In the same section, we shall prove the following positive result.

COROLLARY 2.1. *If* (2.1) *holds and either* (i) $c \equiv 0$, $b(0+) < \infty$, *and* b *is strongly positive, or* (ii)

$$\limsup_{x \to 0+} \frac{\int_0^x b(t)dt}{\int_0^x c(t)dt} < \infty ,$$

then (1.6) *and* (1.7) *hold*.

Thus, in particular, (1.8) holds if $a(t)$ satisfies (1.3) and $-a'(t)$ is convex.

Integration of (1.5) (see also [11], [4]) shows that

$$u(t,\lambda) + \lambda \int_0^t [d + a(t-s)] \int_0^s u(r,\lambda)dr\, ds = 1 ,$$

so that when (1.3) and (2.3i) hold,

$$\int_0^\infty u(t,\lambda)dt = 1/(\lambda \int_0^\infty [d + a(t)]dt)$$

(interpreted as zero if $d + a(t) \notin L^1(R^+)$.)

Thus in Theorem 2.2 we have

$$\int_0^\infty \underset{\sim}{U}(t)dt = \underset{\sim}{L}^{-1}/(\int_0^\infty [d+a(t)]dt) .$$

Detailed statements about the asymptotic nature of $u(t,\lambda)$ as $t \to \infty$ (λ fixed) are given for certain special cases by Levin and Nohel [13, 14], by the second author [5], and by Wong and Wong [24]. For example if $d + a(t) = t^{-\beta}$ $(0 < \infty < 1)$, Corollary 2.1 applies and [5, Cor. 3.3] shows that

$$u(t,\lambda) \sim \frac{C}{\lambda} t^{\beta-2} \quad (t \to \infty) .$$

On the other hand (see Section 4), there is a $C' > 0$ such that

$$\int_0^\infty |u(t,\lambda)|dt \geq C'\lambda^{-1/(2-\beta)} \quad (\lambda > 0) . \tag{2.5}$$

Thus the asymptotic behavior of $u(t,\lambda)$ as $\lambda \to \infty$ is not completely clear. See [14, 9] for further discussion.

Another useful example, where (1.5) can be solved explicitly, is $d + a(t) = e^{-t}$. Then [10] (1.5) reduces to an ordinary differential equation, and

$$(2.6) \qquad u(t,\lambda) = e^{-t/2}(\cos\mu t + \frac{1}{2}\mu^{-1}\sin\mu t) \quad (\lambda \neq \frac{1}{4})$$

$$u(t,\tfrac{1}{4}) = e^{-t/2}(1 + \frac{1}{2}t) \quad ,$$

where $\mu = \frac{1}{2}(4\lambda - 1)^{1/2}$ (λ and μ may be real or complex). For this example, we remark that

(i) $t \to \underset{\sim}{U}(t)$ is not continuous in the norm topology if $\underset{\sim}{L}$ is unbounded, since $\|\underset{\sim}{U}(t) - \underset{\sim}{U}(s)\| \geq |u(t,\lambda) - u(s,\lambda)|$ for λ in the spectrum of $\underset{\sim}{L}$.

(ii) $\int_0^\infty |u(t,\lambda)|dt \geq 1 \quad (\lambda > 0)$. This is proved in [10].

(iii) $u(\cdot,\lambda e^{i\varphi}) \notin L^p(R^+)$ if $p \geq 1$, $e^{i\varphi} \neq 1$, and $\lambda > 0$ is sufficiently large.

Dafermos [2] and Slemrod [21] study equations similar to (1.1) as linear models in viscoelasticity and fluid mechanics respectively. These studies contain no analogue of (1.8).

Our results and methods are closer to those of Friedman and Shinbrot [3], who obtain L^p estimates ($1 \leq p < \infty$) for the resolvent (fundamental solution) $\underset{\sim}{S}(t)$ of

$$(2.7) \qquad \underset{\sim}{y}(t) + \int_0^t h(t-s)\, \underset{\sim}{L}\, \underset{\sim}{y}(s) = \underset{\sim}{F}(t)$$

in Banach space. Formal differentiation of (2.7) yields (1.1) if $h'(t) = d + a(t)$, $h(0) = 0$, $\underset{\sim}{F}' = \underset{\sim}{f}$. For their L^p estimates, Friedman and Shinbrot require at least $h(0) > 0$, $h' \in L^1(R^+)$.

Miller and Wheeler [17] use procedures similar to those of [3] to study the equation

$$(2.8) \qquad \underset{\sim}{y}'(t) = -\underset{\sim}{L}\,\underset{\sim}{y}(t) - \int_0^t a(t-s)\,(\underset{\sim}{L} + \ell\underset{\sim}{I})\,\underset{\sim}{y}(s)ds + \underset{\sim}{f}(t)$$

in Hilbert space. Here $\underset{\sim}{L}$ is self-adjoint and bounded below and has a compact resolvent. Miller and Wheeler give conditions under which the resolvent for (2.8) may be decomposed into an exponential polynomial with finite-dimensional projections as coefficients and a remainder ("residual resolvent") $\underset{\sim}{R}(t)$ with $\|\underset{\sim}{R}(t)\| \in L^p(R^+)$.

The proofs of these results in [3] and [17] use the operational calculus based on contour integrals and estimates such as

$$\int_0^\infty |r(t,\lambda)|^p \, dt \le C|\lambda|^{-\delta} \tag{2.9}$$

($|\arg\lambda| \le (\pi/2) - \varepsilon$), where $\varepsilon, \delta > 0$ and $r(t,\lambda)$ is the solution of a certain scalar equation (analogous to (1.5)) with complex parameter λ. Remarks (ii) and (iii) following (2.6) above show that estimates like (2.9) need not hold for our function $u(t,\lambda)$.

For a broader treatment of existence, uniqueness, and continuous dependence for equations like (1.1) in Banach space, see Miller [16]; further discussion of the resolvent formula (1.9) will also be found in [16].

Finally, we remark that nonlinear versions of (1.1) are under active study [1, 15].

3. <u>Proof of Theorem 2.1</u>. (i). The proof of Theorem 2 of [6], with $a(t)$ replaced by $d + a(t)$ (and the last equalities corrected to read $2V(0) = u^2(0) = 1$), shows that $|u(t,\lambda)| \leq 1$ $(t \in \overline{R}^+, \lambda \in R^+)$, so $\|U(t)\| \leq 1$ and $U(t)L = L\,U(t)$ on $\mathfrak{D}$. Since

$$\|U(t)x - U(s)x\|^2 = \int_\Lambda |u(t,\lambda) - u(s,\lambda)|^2 \, d(E_\lambda x, x) \quad ,$$

the continuity of $u(t,\lambda)$ in t and the dominated convergence theorem imply that $U(t)$ is strongly continuous.

The computations for (ii) are formally the same as those for Theorem 2 of [9], where a and $L\,f$ are continuous on $\overline{R}^+$. To simplify formulas we take $d = 0$ since this does not change the following argument. It is obvious that the function $y(t)$ of (1.9) satisfies (1.2). Let $T(t)$ be the triangle $\{0 \leq r < s \leq t\}$ $(t > 0)$, and let $h(t) = \int_0^t a(s)ds$. Since $L\,U(r)y_0 = U(r)L\,y_0$ is continuous $(r \in \overline{R}^+)$, $a(s-r)L\,U(r)y_0$ is in $L^1(T(t))$ and

$$(3.1) \qquad y_0 - \int_0^t\int_0^s a(s-r)\, L\,U(r)y_0 \, dr\, ds$$

$$= y_0 - \int_0^t h(t-r)\, L\,U(r)y_0 \, dr$$

$$= y_0 - \int_0^t h(t-r)[\lim_{M\to\infty} \int_{\lambda_0}^M \lambda u(r,\lambda)dE_\lambda y_0]dr$$

$$= y_0 - \int_{\lambda_0}^\infty [\lambda \int_0^t h(t-r)\, u(r,\lambda)dr]dE_\lambda\, y_0 \; .$$

The expression in brackets here is just $1 - u(t,\lambda)$, as one sees by integrating (1.5); thus the left-hand side of (3.1) is equal to $U(t)y_0$ and differentiation establishes that

$$(3.2) \qquad \frac{d}{dt}[U(t)y_0] = -\int_0^t a(t-s)\, L\,U(s)y_0 \, ds \quad .$$

We observe next that the strong continuity and uniform boundedness of U ensure that the function

$$a(t-s)\, L\,U(s-r)\, f(r) = a(t-s)\, U(s-r)\, L\, f(r)$$

is strongly measurable on $T(t)$. In view of our hypotheses and [11, Theorem 3.5.4], the following lemma establishes this. (Compare [16, Lemma 2.1].)

Lemma 3.1. If $\underset{\sim}{g}: R^+ \to X$ *belongs to* $B^1(0,t)$, *then the function* $\underset{\sim}{G}(s,r) = \underset{\sim}{U}(s-r)\,\underset{\sim}{g}(r)$ *is strongly measurable on* $T(t)$.

Proof. To simplify notation, take $t = 1$. For each positive integer n, let $\underset{\sim}{U}_{n,j} = \underset{\sim}{U}(j/n)$, $E_{n,j} = ((j-1)/n, j/n]$ $(1 \le j \le n)$. Let $\underset{\sim}{g}_n$ be a sequence of countably-valued functions

$$\underset{\sim}{g}_n(r) = \sum_{k=1}^{\infty} \chi_{n,k}(r)\,\underset{\sim}{g}_{n,k}$$

($\chi_{n,k}$ = the characteristic function of a measurable set $\Omega_{n,k}$) such that $\underset{\sim}{g}_n(r) \to \underset{\sim}{g}(r)$ $(n \to \infty)$ except on a set Z of measure zero. For $(s,r) \in T(1)$, let $j(s,r,n)$ be the integer such that $s-r \in E_{n,j(s,r,n)}$, and let

$$\underset{\sim}{G}_n(s,r) = \underset{\sim}{U}_{n,j(s,r,n)}\,\underset{\sim}{g}_n(r) .$$

Then $\underset{\sim}{G}_n(s,r)$ is measurable and countably valued since $T(1)$ is the union of the measurable sets

$$\{s-r \in E_{n,m}\} \cap \{r \in \Omega_{n,k}\}$$

$(1 \le m \le n,\ 1 \le k < \infty)$, on each of which $\underset{\sim}{G}_n$ is constant. For fixed $(s,r) \in T(1)$, $r \notin Z$,

$$\begin{aligned} &\|\underset{\sim}{G}_n(s,r) - \underset{\sim}{G}(s,r)\| \\ &\quad \le \|\underset{\sim}{U}_{n,j(s,r,n)}\,[\underset{\sim}{g}_n(r) - \underset{\sim}{g}(r)]\| \\ &\qquad + \|[\underset{\sim}{U}_{n,j(s,r,n)} - \underset{\sim}{U}(s-r)]\underset{\sim}{g}(r)\| . \end{aligned}$$

As $n \to \infty$, the first term tends to zero, since $\|U\| \le 1$ and $\underset{\sim}{g}_n(r) \to \underset{\sim}{g}(r)$; by strong continuity, the second term tends to zero as well. Thus $\underset{\sim}{G}(s,r)$ is the limit almost everywhere of countably-valued measurable functions, and the lemma is proved.

Continuing the proof of Theorem 2.1, we note that

$$\begin{aligned} &\int_0^t\int_0^s a(t-s)\,\|\underset{\sim}{U}(s-r)\,\underset{\sim}{L}\underset{\sim}{f}(r)\|\,dr\,ds \\ &\quad \le \int_0^t a(s)ds \int_0^t \|\underset{\sim}{L}\,\underset{\sim}{f}(r)\|dr < \infty , \end{aligned}$$

so $a(t-s)\, \underset{\sim}{U}(s-r)\, \underset{\sim\sim}{Lf}(r) \in B^1(T(t))$. Then, using Fubini's theorem, a change of variable, and the fact that $\underset{\sim}{L}$ is closed, we may compute

$$\int_0^t a(t-s)\underset{\sim}{L} \int_0^s \underset{\sim}{U}(s-r)\ \underset{\sim}{f}(r)dr\ ds \tag{3.3}$$

$$= \int_0^t\int_0^s a(t-s)\ \underset{\sim}{L}\ \underset{\sim}{U}(s-r)\ \underset{\sim}{f}(r)dr\ ds$$

$$= \int_0^t\int_0^{t-r} [a(t-r-s)\ \underset{\sim}{L}\ \underset{\sim}{U}(s)\ \underset{\sim}{f}(r)]ds\ dr$$

$$= -\int_0^t \frac{d}{dt}\ [\underset{\sim}{U}(t-r)\ \underset{\sim}{f}(r)]dr \quad ,$$

where the last step uses (3.2) and $\underset{\sim}{f}(r)$ in place of $\underset{\sim}{y}_0$. It is clear from these equalities that the integrand in the last expression is locally Bochner integrable in (t,r) ; using Fubini's theorem, we see that this expression (and hence the left-hand side (3.3)) is equal to

$$\underset{\sim}{f}(t) - \frac{d}{dt} \int_0^t \underset{\sim}{U}(t-r)\ \underset{\sim}{f}(r)dr \quad .$$

In view of (3.2) this establishes (1.1).

For uniqueness, we pass to the weak, integrated version of (1.1), (1.2) and project on $\underset{\sim}{E}_\lambda \mathcal{X}$; see [9] or [7] for details.

4. Proof of Theorem 2.2. Reduction to two estimates.

We assume without further mention that $d + a(t)$ has been rescaled, if necessary, so that $\Lambda = 1$. The functions a', b', and c' are redefined where necessary so as to be continuous from the left on $\mathbb{R}^+$. We let $A(t) = \int_0^t a(r)dr$.

The proof relies on detailed information about $\hat{a}$ (see (2.2)). See [4, 20] for earlier versions of these ideas.

Lemma 4.1. *Suppose* (1.3) *holds. Then* φ *and* θ *are continuously differentiable on* $\mathbb{R}^+$ *with*

$$\frac{1}{2\sqrt{2}} A(\tfrac{1}{\tau}) \le |\hat{a}(\tau)| \le 4A(\tfrac{1}{\tau}) \qquad (\tau > 0) , \tag{4.1}$$

$$|\hat{a}'(\tau)| \le 40 \int_0^{\frac{1}{\tau}} ra(r)dr \qquad (\tau > 0) , \tag{4.2}$$

$$\frac{1}{5} \int_0^{\frac{1}{\tau}} ra(r)dr \le \theta(\tau) \le 12 \int_0^{\frac{1}{\tau}} ra(r)dr \qquad (\tau > 0) , \tag{4.3}$$

$$-\theta'(\tau) \ge \frac{\tau}{5} \int_0^{\frac{1}{\tau}} r^3 a(r)dr \qquad (\tau > 0) . \tag{4.4}$$

Our proof is adapted from [10, Lemma 2.2]. We exploit the fact that $da'(t)$ is a positive measure on $\mathbb{R}^+$ and adapt the convention, consistent with our choice of $a'(t) = a'(t-)$ that when $0 \le x \le y$ and $f \in L^1(da'(t))$,

$$\int_x^y f(t)\, da'(t) = \int_{[x,y)} f(t)\, da'(t) .$$

Convexity of $a(t)$ implies $a(\frac{t}{2}) - a(t) \ge -\frac{t}{2} a'(t)$, and hence

$$2 \int_0^{\frac{t}{2}} a(r)dr \ge ta(\tfrac{t}{2}) \ge ta(t) - \frac{t^2}{2} a'(t) \ge 0 \qquad (t \ge 0) . \tag{4.5}$$

In particular (4.5) shows that $ta(t) + \frac{t^2}{2}|a'(t)| = o(1)$ $(t \to 0+)$. We also have $ta'(t) = o(1)$ $(t \to \infty)$, as a consequence of (1.3) and

$$\int_T^\infty rda'(r) = a(T) - T\, a'(T) < \infty \qquad (T > 0) . \tag{4.6}$$

Two integrations by parts in (2.2) yield the formula

$$\hat{a}(\tau) = \tau^{-2} \int_0^\infty (1 - i\tau r - e^{-i\tau r})da'(r) \qquad (\tau > 0) , \tag{4.7}$$

where (4.5) and (4.6) assure vanishing of the boundary terms and absolute convergence of the integral.

Following [20], we let $J(u) = iu(1-e^{iu}) - 2(1 + iu - e^{iu})$; then

$$|J(u)| \le \frac{1}{4} u^3 (0 \le u \le 1), \quad \text{and} \quad |J(u)| \le 2(u + 2) \qquad (u \ge 0) . \tag{4.8}$$

(4.8), combined with Fubini's Theorem, justifies differentiation of (4.7) and gives us

$$\hat{a}'(\tau) = \tau^{-3} \int_0^\infty J(-\tau r) da'(r) \qquad (\tau > 0) . \tag{4.9}$$

The inequalities (4.1) and (4.2) now follow as in [20].

From (4.7) we have

$$\theta(\tau) = \tau^{-2} \int_0^\infty rK(\tau r)\, da'(r) \qquad (\tau > 0) , \tag{4.10}$$

with

$$K(u) = 1 - \frac{\sin u}{u} \qquad (u > 0) .$$

Note that

$$\begin{cases} K(u) \ge \frac{u^2}{6} - \frac{u^2}{120} \ge \frac{u^2}{30} & (0 \le u \le 1) \\ K(u) \ge 1 - \max\{\sin 1, \frac{2}{\pi}\} \ge \frac{1}{10} & (u \ge 1) . \end{cases}$$

Therefore,

$$\theta(\tau) \ge \frac{1}{30} \int_0^{\frac{1}{\tau}} r^3\, da'(r) + \frac{1}{10\tau^2} \int_{\frac{1}{\tau}}^\infty rda'(r) \quad , \qquad (\tau > 0) . \tag{4.11}$$

The relations

$$\begin{cases} \int_{\frac{1}{\tau}}^\infty rda'(r) = -\frac{1}{\tau} a'(\frac{1}{\tau}) + a(\frac{1}{\tau}) \\ \int_0^{\frac{1}{\tau}} r^3 da'(r) = \frac{1}{\tau^3} a'(\frac{1}{\tau}) - \frac{3}{\tau^2} a(\frac{1}{\tau}) + 6 \int_0^{\frac{1}{\tau}} ra(r)dr \quad , \end{cases} \tag{4.12}$$

along with (4.11), give us the first inequality in (4.3). The second inequality follows from the estimates

$$|K(u)| \le 2u^2 \quad (0 \le u \le 1) \quad \text{and} \quad |K(u)| \le 2 \qquad (u > 0) \quad ,$$

which, along with (4.12), yield

$$\theta(\tau) \le 2 \int_0^{\frac{1}{\tau}} r^3 \, da'(r) + 2\tau^{-2} \int_{\frac{1}{\tau}}^{\infty} r\,da'(r) \le 12 \int_0^{\frac{1}{\tau}} ra(r)dr \ .$$

To prove (4.4) we differentiate (4.10), which yields

(4.13) $$-\tau^4 \, \theta'(\tau) = \int_0^{\infty} H(\tau r) \, da'(r) \quad , \qquad (\tau > 0) \quad ,$$

where

$$H(u) = 3(u - \sin u) - u(1 - \cos u) \ .$$

Then

(4.14) $$\begin{cases} H^{(j)}(0) = 0 & (j = 0,1,2,3,4) \\ H^{(5)}(0) = 2 & \\ -4 \le -3\sin u - u\cos u = H^{(6)}(u) \le 0 & (0 \le u \le 1) \end{cases}$$

so

(4.15) $$H(u) \ge \frac{2u^5}{5!} - \frac{4u^6}{6!} = \frac{u^5}{180}(3-u) \ge \frac{u^5}{100} \qquad (0 \le u \le 1) \ ,$$

and

(4.16) $$H'(u) \ge \frac{2u^4}{4!} - \frac{4u^5}{5!} = \frac{u^4}{60}(5-2u) \ge \frac{u^4}{20} \qquad (0 \le u \le 1) \ .$$

If we can show that $H''(u) \ge 0$ $(1 \le u \le 4)$, it will then follow from (4.15) and (4.16) that

(4.17) $$H(u) \ge \frac{1}{100} + \frac{1}{20}(u-1) = \frac{1}{100}(5u-4) \qquad (1 \le u \le 4) \ .$$

But $H^{(3)}(u) = u \sin u \ge 0$ $(0 \le u \le \pi)$; by (4.14), we conclude that $H''(u) \ge 0 (0 \le u \le \pi)$. Then since $H^{(3)}(u) \le 0$ $(\pi \le u \le \frac{4\pi}{3})$, $H^{(2)}(u) \ge H^{(2)}(\frac{4\pi}{3}) > 0$ $(\pi \le u \le 4)$, so (4.17) follows.

It is easy to see that

(4.18) $$H(u) \ge u-3 \qquad (u \ge 4) \ ,$$

and thus (4.13), along with (4.15), (4.17), and (4.18), gives

$$-\tau^4\,\theta'(\tau) \geq \frac{\tau^5}{100}\int_0^{\frac{1}{\tau}} r^5\,da'(r) + \frac{1}{100}\int_{\frac{1}{\tau}}^{\frac{4}{\tau}} (5\tau r - 1)\,da'(r) + \int_{\frac{4}{\tau}}^{\infty} (\tau r - 3)\,da'(r)\ , \tag{4.19}$$

and

$$\left\{\begin{array}{l} \dfrac{\tau^5}{100}\displaystyle\int_0^{\frac{1}{\tau}} r^5\,da'(r) = \frac{1}{100}\,a'(\tfrac{1}{\tau}) - \frac{\tau}{20}\,a(\tfrac{1}{\tau}) + \frac{\tau^5}{5}\int_0^{\frac{1}{\tau}} r^3\,a(r)\,dr \\[2ex] \dfrac{1}{100}\displaystyle\int_{\frac{1}{\tau}}^{\frac{4}{\tau}} (5\tau r - 4)\,da'(r) = \frac{4}{25}\,a'(\tfrac{4}{\tau}) - \frac{1}{100}\,a'(\tfrac{1}{\tau}) - \frac{\tau}{20}\,a(\tfrac{4}{\tau}) + \frac{\tau}{20}\,a(\tfrac{1}{\tau}) \\[2ex] \displaystyle\int_{\frac{4}{\tau}}^{\infty} (\tau r - 3)\,da'(r) = -a'(\tfrac{4}{\tau}) + \tau a(\tfrac{4}{\tau})\ . \end{array}\right. \tag{4.20}$$

Combining (4.19) and (4.20) we obtain (4.4). This completes the proof of Lemma 4.1.

Using (4.3) and (4.4) we see that the equation

$$\theta(\omega) + \frac{d}{\omega^2} = \frac{1}{\lambda} \tag{4.21}$$

defines a strictly increasing, continuously differentiable function $\omega = \tilde{\omega}(\lambda)$ on a subinterval (λ_0,∞) of $\mathbb{R}^+$ with $\tilde{\omega}(\lambda) \to \infty$ as $\lambda \to \infty$. This provides the missing step in the proof of Theorem 2.2(iii) outlined just above the statement of the theorem.

Fix $t_1 > 0$ such that $a(t_1) > 0$ and let $\rho = 6t_1^{-1}$. Set $\omega(\lambda) = \max\{\rho,\tilde{\omega}(\lambda)\}$ if $\tilde{\omega}(\lambda)$ is defined, $\omega(\lambda) = \rho$ otherwise.

(4.3) and (4.21) yield

$$\frac{1}{\lambda} \geq \theta(\omega) \geq \frac{1}{5}\int_0^{\frac{1}{\omega}} r\,a(r)\,dr\ , \tag{4.22}$$

and

$$\int_0^{\frac{1}{\omega}} r\,a(r)\,dr \geq \frac{1}{2\omega^2}\,a(\tfrac{1}{\omega}) \geq \frac{a(t_1)}{2\omega^2}\ . \tag{4.23}$$

In particular, (4.22) and (4.23) show that

$$(4.24)\qquad \omega^2(\lambda) \geq \frac{a(t_1)}{10}\lambda \ .$$

In [10] it was also shown that when $a(0+) < \infty$, $\omega^2(\lambda)$ is bounded above by some constant times λ. Such an estimate is not available to us when $a(0+) = \infty$, and this causes the principal new difficulties in the proof of Theorem 2.2.

When $a(t) = t^{-\beta}$ $(0 < \beta < 1,\ t > 0)$, direct computations show that $\omega(\lambda) = K\lambda^{\frac{1}{2-\beta}}$ $(K = K(\beta) > 0)$. Now the inequality $\int_0^\infty |u(t,\lambda)|dt \geq |\hat{u}(\omega(\lambda))| \geq \theta(\omega(\lambda))/\dot{\varphi}(\omega(\lambda))$ can be used to derive the estimate (2.5).

From (4.3), (4.21), and (4.23) we obtain

$$(4.25)\qquad \frac{1}{\lambda} \leq 12 \int_0^{\frac{1}{\omega}} ra(r)dr + \frac{d}{\omega^2} \leq (12 + \frac{2d}{a(t_1)}) \int_0^{\frac{1}{\omega}} ra(r)dr$$

whenever

$$\lambda \geq \lambda_0 = \max\{(\theta(\rho) + \frac{d}{\rho^2})^{-1},\ 1\} \ .$$

On the other hand when $1 \leq \lambda \leq \lambda_0$, $\omega(\lambda) = \rho$ and we have

$$(4.26)\qquad \frac{1}{\lambda} \leq 1 \leq \lambda_0(12 + \frac{2d}{a(t_1)}) \int_0^{\frac{1}{\rho}} ra(r)dr \ .$$

Then, combining (4.22), (4.25) and (4.26) we find that

$$(4.27)\qquad \frac{1}{5}\int_0^{\frac{1}{\omega}} ra(r)dr \leq \frac{1}{\lambda} \leq C \int_0^{\frac{1}{\omega}} ra(r)dr \qquad (\lambda \geq 1) \ ,$$

where $C = [12 + \frac{2d}{a(t_1)}]\lambda_0$.

Define

$$(4.28)\qquad \begin{cases} D(\tau) \equiv D(\tau,\infty) = \hat{a}(\tau) - id\tau^{-1} \\ \\ D(\tau,\lambda) = D(\tau) + i\tau\lambda^{-1} = \varphi(\tau) + i\tau(\lambda^{-1} - \theta(\tau) - d\tau^{-2}) \end{cases}$$

$$(1 \leq \lambda \leq \infty,\ \tau > 0) \ .$$

If (2.3i) holds then $|D(\tau,\lambda)| \geq \varphi(\tau) > 0$ $(\tau > 0,\ 1 \leq \lambda \leq \infty)$ and [4] gives the representation

(4.29) $$\pi u(t,\lambda) = \frac{1}{\lambda}\int_0^\infty \mathrm{Re}\,\{\frac{e^{i\tau t}}{D(\tau,\lambda)}\}\,dt \qquad (0 < t < \infty\ ,\ 0 < \lambda < \infty)\ .$$

(The integral is improper at $\tau = \infty$; by (1.3) and (4.1) $[D(\tau,\lambda)]^{-1}$ is continuous at $\tau = 0$ and for every $\tau > 0$.) Moreover the result of Shea and Wainger [20] shows that

(4.30) $$\int_0^\infty |u(t,\lambda)|dt < \infty \qquad (\lambda > 0)\ ,$$

and in [6] it was shown that (1.3) implies

(4.31) $$\sup_{1\leq\lambda\leq\infty} |u(t,\lambda)| = 1\ .$$

For the remainder of the proof, unless noted otherwise, we assume that (1.3), (2.1), and (2.3) hold. Return to (4.29) and integrate by parts. There results the formula

(4.32) $$\pi u(t,\lambda) = \mathrm{Re}\,\{\frac{1}{it\lambda}\int_0^\infty e^{i\tau t}\frac{D_\tau(\tau,\lambda)}{[D(\tau,\lambda)]^2}\,d\tau\} \qquad (t > 0,\ \lambda > 0)\ .$$

Relations (4.1) and (4.2) show that the boundary terms vanish and that the integral converges absolutely when $d \neq 0$. Absolute convergence of the integral when $d = 0$ is assured by an estimate of Shea and Wainger [20, pp. 322-323], namely

(4.33) $$\int_0^\rho \frac{\int_0^{\frac{1}{\tau}} ra(r)dr}{(\int_0^{\frac{1}{\tau}} a(r)dr)^2}\,d\tau < \infty\ .$$

Note that

$$\frac{1}{D(\tau,\lambda)} - \frac{1}{D(\tau)} = \frac{-i\tau\lambda^{-1}}{D(\tau,\lambda)D(\tau)}\ .$$

Then

(4.34) $$\frac{D_\tau(\tau,\lambda)}{[D(\tau,\lambda)]^2} = \frac{D_\tau(\tau,\lambda)}{[D(\tau)]^2}\,[1 - \frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}]^2$$

$$= \frac{[D'(\tau) + i\lambda^{-1}]}{[D(\tau)]^2}\,[1 - \frac{2i\tau\lambda^{-1}}{D(\tau)}] - \frac{\tau^2 D_\tau(\tau,\lambda)}{\lambda^2[D(\tau)]^2 D(\tau,\lambda)}\,[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}]\ .$$

Define

$$
(4.35)\quad \begin{cases}
u_1(t) = \frac{1}{t} \int_0^\rho e^{i\tau t} \frac{D'(\tau)}{[D(\tau)]^2} d\tau & (t > 0) , \\[2ex]
u_2(t) = \frac{1}{t} \int_0^\rho \frac{e^{i\tau t}}{[D(\tau)]^2} [1 - \frac{2\,D'(\tau)}{D(\tau)}] d\tau & (t > 0) , \\[2ex]
u_3(t) = \frac{1}{t} \int_0^\rho e^{i\tau t} \frac{2\tau}{[D(\tau)]^3} d\tau & (t > 0) , \\[2ex]
u_4(t,\lambda) = \frac{-1}{\lambda^3 t} \int_0^\rho e^{i\tau t} \frac{\tau^2 D_\tau(\tau,\lambda)}{[D(\tau)]^3\, D(\tau,\lambda)} \left[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right] d\tau & (t > 0) , \\[2ex]
u_5(t,\lambda) = \frac{1}{t\lambda} \int_\rho^\infty e^{i\tau t} \frac{D_\tau(\tau,\lambda)}{[D(\tau,\lambda)]^2} d\tau & (t > 0) ,
\end{cases}
$$

Referring to (4.32) and (4.34) we see that

$$(4.36)\qquad u(t,\lambda) = \mathrm{Im}\{\lambda^{-1}u_1(t) + i\lambda^{-2}u_2(t) + \lambda^{-3}u_3(t) + u_4(t,\lambda) + u_5(t,\lambda)\} .$$

In section 6 we will show that if (1.3), (2.1) and (2.3) hold then

$$(4.37)\qquad |u_j(t,\lambda)| \le Mq(t) \qquad (t > 0,\ 1 \le \lambda < \infty,\quad j = 4, 5) ,$$

where, now and henceforth, M (or M_j) denotes a positive constant independent of λ whose value may change from line to line, and

$$q(t) = t^{-2}\int_0^t b(r)dr + t^{-2} + t^{-1}\, b(t) - b'(t), \qquad (t > 0) .$$

The assumption (2.1i) combined with an integration by parts shows that

$$\int_1^T t^{-2}\int_0^t b(r)dr\, dt = -T^{-1}\int_0^T b(t)dt + \int_0^1 b(t)dt + \int_1^T \frac{b(t)}{t}\, dt$$

from which it follows easily that $q(t) \in L^1(1,\infty)$.

In view of (4.30), (4.36), and (4.37) we find that $u_j(t) \in L^1(1,\infty)$ $(j = 1,2,3)$, and this, along with (4.31), implies that

$$\int_0^\infty \sup_{1 \le \lambda < \infty} |u(t,\lambda)| dt < \infty \quad .$$

This is the assertion of Theorem 2.2(ii).

Similar estimates which hold when (1.3) and (2.3) hold, but without the assumptions of (2.1), can then be used to show that

$$(4.38) \qquad |u_1(t)| + |u_2(t)| + |u_3(t)| + |u_4(t,\lambda)| + |u_5(t,\lambda)| \le \frac{M}{t} \qquad (1 \le \lambda < \infty,\ 0 < t) \ ,$$

from which Theorem 2.2(i) follows. (Theorem 2.2(iii) has already been proved above.)

5. Two more lemmas - some important estimates

We would like to integrate by parts in the formulas for u_4 and u_5 in order to bring out another factor of t^{-1}. Hypothesis (2.1i) appears to be too weak to permit this, so we separate $\hat{b}'$ into a "small" part and a differentiable part.

We again let $J(u) = iu(1 - e^{iu}) - 2(1 - iu - e^{iu})$ and define

$$\beta^0(t,\tau) = \tau^{-3}\int_0^t J(-\tau r)db'(r) \qquad (0 < t < \infty,\ 0 < \tau < \infty) \quad ,$$

(5.1)

$$\beta^\infty(t,\tau) = \tau^{-3}\int_t^\infty J(-\tau r)\, db'(r) \qquad (0 < t < \infty,\ 0 < \tau < \infty) \quad .$$

Observe that $\hat{b}(\tau)$ and $\hat{b}'(\tau)$ are given by expressions analogous to (4.7) and (4.9) respectively so that, in particular

$$\text{(5.2)} \qquad \hat{b}'(\tau) = \beta^0(t,\tau) + \beta^\infty(t,\tau) \qquad (0 < t < \infty,\ 0 < \tau < \infty) \quad .$$

Lemma 5.1. If (1.3) and (2.1) hold then $\hat{c}(\tau)$ is twice continuously differentiable, $\frac{\partial\beta^0}{\partial\tau}(t,\tau)$ exists and is continuous $(t > 0, \tau > 0)$, and

$$\text{(5.3)} \qquad |\hat{c}''(\tau)| \le 6000\int_0^{\frac{1}{\tau}} r^2\, c(r)dr \qquad (\tau > 0) \quad ,$$

$$\text{(5.4)} \qquad |\beta^\infty(t,\tau)| \le 40\tau^{-2}(b(t) - tb'(t)) \qquad (\tau > 0,\ t > 0) \quad ,$$

$$\text{(5.5)} \qquad \left|\frac{\partial\beta^0}{\partial\tau}(t,\tau)\right| \le 500\tau^{-2}\int_0^t b(r)dr \qquad (\tau > 0,\ t > 0) \quad ,$$

$$\text{(5.6)} \quad \text{(i)} \qquad |\beta^0(t,\tau)| \le 40\int_0^{\frac{1}{\tau}} rb(r)dr \qquad (\tau > 0,\ t > 0) \quad ,$$

$$\text{(ii)} \qquad |\hat{c}'(\tau)| \le 40\int_0^{\frac{1}{\tau}} rc(r)dr \qquad (\tau > 0) \quad .$$

Proof. Three integrations by parts show that

$$\text{(5.7)} \qquad \hat{c}(\tau) = -i\tau^{-3}\int_0^\infty \left(1 - i\tau r + \frac{(i\tau r)^2}{2} - e^{-i\tau r}\right)dc''(r) \quad ,$$

where we use

$$rc(r) + r^2|c'(r)| + r^3\, c''(r-) \to 0 \qquad (r \to 0+)$$

and

$$c(r) + r\,|c'(r)| + r^2\, c''(r-) \to 0 \qquad (r \to \infty) \quad ,$$

which are consequences of (2.1) and which assure that the various boundary terms vanish. Note that $dc''(r)$ is a negative measure; three differentiations of (5.7) yield

$$\hat{c}''(\tau) = i\tau^{-5}\int_0^\infty K(-\tau r)dc''(r) \qquad (\tau > 0) \quad ,$$

where

$$K(u) = -12(1 + iu + \frac{(iu)^2}{2} - e^{iu}) + 6iu(1 + iu - e^{iu}) + u^2(1-e^{iu}) \quad .$$

The remainder of the proof of (5.3) now follows as in [10, Lemma 6.1(ii)].

To obtain (5.4) we first consider the case where $\tau t \geq 1$. (4.8) and (5.1) give us

$$|\beta^\infty(t,\tau)| \leq 2\tau^{-3}\int_t^\infty (\tau r+2)db'(r) \leq 2\tau^{-2}(b(t) - 3tb'(t))$$
$$\leq 40\tau^{-2}(b(t) - tb'(t)) \quad .$$

On the other hand if $\tau t < 1$ then (4.8) yields

$$|\beta^\infty(t,\tau)| \leq 6\int_t^{\frac{1}{\tau}} r^3 db'(r) + 2\tau^{-3}\int_{\frac{1}{\tau}}^\infty (\tau r + 2)db'(r)$$
$$= 6[\tau^{-3}b'(\tfrac{1}{\tau}) - t^3b'(t) - 3\tau^{-2}b(\tfrac{1}{\tau}) + 3t^2b(t) + 6\int_t^{\frac{1}{\tau}} rb(r)dr]$$
$$+ 2\tau^{-2}b(\tfrac{1}{\tau}) - 6\,\tau^{-3}b'(\tfrac{1}{\tau})$$
$$\leq -6t^3b'(t) + 18t^2b(t) + 36b(t)\int_t^{\frac{1}{\tau}} rdr$$
$$\leq 40\tau^{-2}(b(t) - tb'(t)) \quad ,$$

thus giving us (5.4) in both cases.

Differentiation of (5.1) yields

$$\frac{\partial\beta^0}{\partial\tau}(t,\tau) = \tau^{-4}\int_0^t K(-\tau r)db'(r) \qquad (\tau > 0, t > 0) \quad ,$$

where $K(u) = 6(1 + iu - e^{iu}) - 4iu(1 - e^{iu}) + u^2e^{iu}$, and we have

$$|K(u)| \leq u^4 \ (0 \leq u \leq 1), \quad \text{and} \quad |K(u)| \leq 20(1 + u^2) \qquad (u > 0) \ .$$

If $\tau t > 1$,

$$\left|\frac{\partial\beta^0}{\partial\tau}(t,\tau)\right| \le 40\int_0^{\frac{1}{\tau}} r^4\, db'(r) + 20\tau^{-4}\int_{\frac{1}{\tau}}^{t} (\tau^2 r^2 + 1)\, db'(r)$$

$$= 40[\tau^{-4} b'(\tfrac{1}{\tau}) - 4\tau^{-3} b(\tfrac{1}{\tau}) + 12\int_0^{\frac{1}{\tau}} r^2 b(r)dr]$$

$$+ 20\tau^{-2}[t^2 b'(t) - \tau^{-2} b'(\tfrac{1}{\tau}) - 2tb(t) + 2\tau^{-1} b(\tfrac{1}{\tau}) + 2\int_{\frac{1}{\tau}}^{t} b(r)dr]$$

$$+ 20\tau^{-4}[b'(t) - b'(\tfrac{1}{\tau})]$$

$$\le 40\tau^{-2}\int_{\frac{1}{\tau}}^{t} b(r)dr + 480\int_0^{\frac{1}{\tau}} r^2 b(r)dr$$

$$\le 500\tau^{-2}\int_0^t b(r)dr .$$

If $\tau t < 1$ then

$$\left|\frac{\partial\beta^0}{\partial\tau}(t,\tau)\right| \le \int_0^t r^4\, db'(r) \le 12\int_0^t r^2 b(r)dr \le \frac{12}{\tau^2}\int_0^t b(r)dr ,$$

so we have established (5.5) in both cases. (5.6) is obtained in the same way as (4.2). This completes the proof of Lemma 5.1.

In order to obtain estimates on the size of u_5 we will need lower bounds on $D(\tau,\lambda)$.

We use $\omega = \omega(\lambda)$ as defined in Section 4.

Lemma 5.2. If (1.3) holds, then

$$|D(\tau,\lambda)| \ge M\tau\int_0^{\frac{1}{\tau}} ra(r)dr \qquad (\tfrac{\rho}{2} \le \tau \le \tfrac{\omega}{2}) , \tag{5.8}$$

$$|D(\tau,\lambda)| \ge M\frac{|\tau-\omega|}{\lambda} \qquad (\tau \ge \tfrac{\omega}{2}) . \tag{5.9}$$

Proof. The estimates (4.3), (4.4), and (4.27) will be exploited throughout without explicit mention.

For $\frac{\rho}{2} \leq \tau < \omega = \rho$,

$$\mathrm{Re}\, D(\tau,\lambda) = \varphi(\tau) \geq M \geq M_1 \tau \int_0^{\frac{1}{\tau}} r a(r) dr \geq M_2 \frac{|\tau-\omega|}{\lambda}$$

which establishes (5.8) and (5.9) in this trivial case.

In all other cases we start with

$$-\theta'(\tau) \geq \frac{\tau}{5} \int_0^{\frac{1}{\tau}} r^3 a(r) dr \geq \frac{1}{80\tau^3} a(\tfrac{1}{\tau}) \quad , \tag{5.10}$$

and use both parts of this estimate to obtain

$$\begin{aligned} |\mathrm{Im}\, D(\tau,\lambda)| &= \tau\left|\frac{1}{\lambda} - \frac{d}{\tau^2} - \theta(\tau)\right| \geq \tau\left|\frac{1}{\lambda} - \frac{d}{\omega^2} - \theta(\tau)\right| \\ &\geq \tau|\theta(\omega) - \theta(\tau)| = \tau\left|\int_\omega^\tau \theta'(s) ds\right| \\ &\geq \frac{\tau}{10} \left|\int_\omega^\tau s \int_0^{\frac{1}{s}} r^3 a(r) dr ds\right| + \frac{\tau}{160} \left|\int_\omega^\tau s^{-3} a(\tfrac{1}{s}) ds\right| \\ &\geq \frac{\tau}{10} \left|\int_0^{\frac{1}{\tau}} r^3 a(r) \int_\omega^\tau s ds\, dr\right| + \frac{\tau}{160} \left|\int_\omega^\tau s^{-3} a(\tfrac{1}{s}) ds\right| \\ &= \frac{\tau|\tau-\omega|(\tau+\omega)}{20} \int_0^{\frac{1}{\tau}} r^3 a(r) dr + \frac{\tau}{160}\left|\int_{\frac{1}{\tau}}^{\frac{1}{\omega}} r\, a(r) dr\right| . \end{aligned} \tag{5.11}$$

In the last step of this computation we used Fubini's Theorem on the first integral and a change of variable $(r = \frac{1}{s})$ on the second. Then

$$\begin{aligned} \frac{\tau|\tau-\omega|(\tau+\omega)}{20} \int_0^{\frac{1}{\tau}} r^3 a(r) dr &\geq \frac{\tau^2|\tau-\omega|}{20} \left[\int_0^{\frac{1}{4\tau}} + \int_{\frac{1}{4\tau}}^{\frac{1}{2\tau}}\right] r^3 a(r) dr \\ &\geq \frac{|\tau-\omega|}{6000} \tau^{-2} (a(\tfrac{1}{4\tau}) + a(\tfrac{1}{2\tau})) . \end{aligned} \tag{5.12}$$

Define $f(t) = ta(t) - \frac{1}{2} t^2 a'(t)$ $(t > 0)$. Observe that $f(t)$ is nonnegative, left continuous (by our convention $a'(t) = a'(t^-)$), and satisfies

(5.13) $$ta(t) \le f(t) \le \int_0^t a(r)dr \qquad (t > 0) .$$

Here the first inequality is immediate from the definition and the second is obtained by integrating by parts twice in the inequality $\frac{1}{2}\int_0^t r^2\, da'(r) \ge 0$. From (4.5) we also have

(5.14) $$f(t) \le ta(\tfrac{t}{2}) \qquad (t > 0) ,$$

so that $f(0+) = 0$ and f is bounded on $[0,\frac{1}{\tau}]$ for every $\tau > 0$. Let $S(\tau) = \sup_{0<x<\frac{1}{\tau}} \{f(x)\}$, and for each $\tau > 0$ choose $\delta = \delta(\tau) \in (0,1]$ so that $f(\frac{\delta}{\tau}) \ge \frac{1}{2}S(\tau)$.

The proof now splits into three cases.

Case 1. If $1 \ge \delta(\tau) \ge \frac{1}{2}$ then (5.14) and (5.13) imply

(5.15) $$\frac{|\tau-\omega|}{6000\tau^2}\, a(\tfrac{1}{4\tau}) \ge \frac{|\tau-\omega|}{6000\,\tau}\left[\frac{\delta}{\tau}\, a(\tfrac{\delta}{2\tau})\right] \ge \frac{|\tau-\omega|}{6000\tau}\, f(\tfrac{\delta}{\tau})$$

$$\ge \frac{|\tau-\omega|}{12000\tau}\, S(\tau) \ge \frac{|\tau-\omega|}{12000}\, (\frac{1}{\tau} \cdot \sup_{0<x<\frac{1}{\tau}} \{x\, a(x)\})$$

$$\ge \frac{|\tau-\omega|}{12000} \int_0^{\frac{1}{\tau}} r\, a(r)dr .$$

If $\frac{\rho}{2} \le \tau \le \frac{\omega}{2}$ then $|\omega - \tau| \ge \frac{1}{2}\omega \ge \tau$ so (5.11), (5.12), and (5.15) combine to give (5.8).

On the other hand, if $\tau \ge \frac{1}{2}\omega$ then (5.11), (5.12), (5.15) and (4.27) yield

$$|\mathrm{Im}\, D(\tau,\lambda)| \ge \frac{|\tau-\omega|}{12000}\int_0^{\frac{1}{\tau}} ra(r)dr + \frac{\tau}{160}\, \left|\int_{\frac{1}{\tau}}^{\frac{1}{\omega}} ra(r)dr\right|$$

$$\ge \frac{|\tau-\omega|}{12000}\int_0^{\frac{1}{\omega}} ra(r)dr \ge M\,\frac{|\tau-\omega|}{\lambda} ,$$

which is (5.9).

Case 2. If $0 < \delta(\tau) < \frac{1}{2}$ and $f(\frac{1}{\tau}) \ge \frac{1}{2}\, f(\frac{\delta}{\tau})$ then, again using (5.14), we obtain

(5.16) $$\frac{|\tau-\omega|}{6000\tau^2}\, a(\frac{1}{2\tau}) \geq \frac{|\tau-\omega|}{6000\tau}\, f(\frac{1}{\tau}) \geq \frac{|\tau-\omega|}{12000\tau}\, f(\frac{\delta}{\tau})$$

$$\geq \frac{|\tau-\omega|}{24000}\, S(\tau) \geq \frac{|\tau-\omega|}{24000}\, (\frac{1}{\tau} \sup_{0<x<\frac{1}{\tau}} \{xa(x)\})$$

$$\geq \frac{|\tau-\omega|}{24000} \int_0^{\frac{1}{\tau}} ra(r)dr \quad .$$

Combining (5.11), (5.12), and (5.16) we complete the computations as in Case 1 for both (5.8) and (5.9).

Case 3. If $0 < \delta(\tau) < \frac{1}{2}$ and $f(\frac{1}{\tau}) \leq \frac{1}{2} f(\frac{\delta}{\tau})$ then we use (4.7) along with the estimate $1-\cos x \geq \frac{x^2}{4}$ $(0 < x < 1)$, and then apply (5.13) to obtain

(5.17) $$2 \operatorname{Re} D(\tau,\lambda) = 2\varphi(\tau) = 2\tau^{-2} \int_0^{\infty} (1 - \cos\tau r)\, da'(r)$$

$$\geq \frac{1}{2} \int_0^{\frac{1}{\tau}} r^2\, da'(r) = \int_0^{\frac{1}{\tau}} a(r)dr - f(\frac{1}{\tau})$$

$$= \tau \int_0^{\frac{1}{\tau}} ra(r)dr + \int_0^{\frac{1}{\tau}} (1 - \tau r)\, a(r)dr - f(\frac{1}{\tau})$$

$$\geq \tau \int_0^{\frac{1}{\tau}} ra(r)dr + (1 - \delta)\int_0^{\frac{\delta}{\tau}} a(r)dr - f(\frac{1}{\tau})$$

$$\geq \tau \int_0^{\frac{1}{\tau}} ra(r)dr + \frac{1}{2} f(\frac{\delta}{\tau}) - f(\frac{1}{\tau})$$

$$\geq \tau \int_0^{\frac{1}{\tau}} ra(r)dr \; .$$

This is (5.8).

When $\tau \geq \frac{\omega}{2}$, (5.11), (5.17), and (4.27) combine to yield

(5.18) $$\sqrt{2}\, |D(\tau,\lambda)| \geq \operatorname{Re} D(\tau,\lambda) + |\operatorname{Im} D(\tau,\lambda)|$$

$$\geq \frac{\tau}{2} \int_0^{\frac{1}{\tau}} ra(r)dr + \frac{\tau}{160} \left|\int_{\frac{1}{\tau}}^{\frac{1}{\omega}} ra(r)dr\right| \geq \frac{|\tau-\omega|}{160} \int_0^{\frac{1}{\omega}} ra(r)dr$$

$$\geq M\, \frac{|\tau-\omega|}{\lambda} \; ,$$

which is (5.9). This completes the proof of Lemma 5.2.

6. Proof of estimate (4.37). Define $\Delta(t,\tau) = \beta^0(t,\tau) + \hat{c}'(\tau) + \dfrac{id}{\tau^2}$ $(\tau > 0, t > 0)$,

and let

$$\mu_{41}(t,\lambda) = \frac{1}{\lambda^3 t}\int_0^\rho e^{i\tau t}\,\frac{\tau^2(\Delta(t,\tau)+i\lambda^{-1})}{[D(\tau)]^3\,D(\tau,\lambda)}\left[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right]d\tau \qquad (t > 0),$$

$$\mu_{42}(t,\lambda) = \frac{1}{\lambda^3 t}\int_0^\rho e^{i\tau t}\,\frac{\tau^2\beta^\infty(t,\tau)}{[D(\tau)]^3\,D(\tau,\lambda)}\left[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right]d\tau \qquad (t > 0),$$

$$\mu_{51}(t,\lambda) = \frac{1}{\lambda t}\int_\rho^\infty e^{i\tau t}\,\frac{(\Delta(t,\tau)+i\lambda^{-1})}{[D(\tau,\lambda)]^2}\,d\tau \qquad (t > 0),$$

$$\mu_{52}(t,\lambda) = \frac{1}{\lambda t}\int_\rho^\infty e^{i\tau t}\,\frac{\beta^\infty(t,\tau)}{[D(\tau,\lambda)]^2}\,d\tau \qquad (t > 0).$$

Thus $u_j(t,\lambda) = \mu_{j1}(t,\lambda) + \mu_{j2}(t,\lambda)$ $(t > 0,\ 1 \le \lambda < \infty,\ j = 4,5)$. We now integrate by parts on μ_{41} and μ_{51} in order to bring out another factor of t^{-1}. This gives us

$$(6.1)\qquad -i\lambda^3 t^2\mu_{41}(t,\lambda) = -\frac{e^{i\rho t}\,\rho^2(\Delta(t,\rho) + i\lambda^{-1})}{D^3(\rho)\,D(\rho,\lambda)}\left[\frac{2}{D(\rho)} + \frac{1}{D(\rho,\lambda)}\right]$$

$$+\int_0^\rho e^{i\tau t}\left[\frac{2\tau(\Delta(t,\tau) + i\lambda^{-1}) + \tau^2\Delta_\tau(t,\tau)}{D^3(\tau)\,D(\tau,\lambda)}\left(\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right)\right.$$

$$\left.-\frac{\tau^2(\Delta(t,\tau) + i\lambda^{-1})}{D^3(\tau)\,D(\tau,\lambda)}\left(\frac{8D'(\tau)}{D^2(\tau)} + \frac{5D'(\tau) + 2i\lambda^{-1}}{D(\tau)\,D(\tau,\lambda)} + \frac{2D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right)\right]d\tau$$

and

$$(6.2)\qquad -i\lambda t^2\,\mu_{51}(t,\lambda) = \frac{e^{i\rho t}(\Delta(t,\rho) + i\lambda^{-1})}{D^2(\rho,\lambda)}$$

$$+\int_\rho^\infty e^{i\tau t}\left[\frac{\Delta_\tau(t,\tau)}{D^2(\tau,\lambda)} - \frac{2(\Delta(t,\tau) + i\lambda^{-1})D_\tau(\tau,\lambda)}{D^3(\tau,\lambda)}\right]d\tau .$$

In (6.1) the estimates of Lemmas 4.1 and 5.1 assure absolute convergence of the integral and vanishing boundary terms at $\tau = 0$. In (6.2) absolute convergence and vanishing boundary terms at $\tau = \infty$ is a consequence of Lemmas 5.1 and 5.2 along with (2.3).

Inequalities (4.1) and (2.3i) imply that

$$\inf_{0<\tau<\rho} \inf_{1\leq\lambda\leq\infty} |D(\tau,\lambda)| \equiv \gamma > 0 \ .$$

From (4.2) and Lemma 5.1 we find the estimates

$$(6.3)\quad \begin{cases} \max\{\tau|D_\tau(\tau,\lambda)|,\ \tau|\Delta(t,\tau) + i\lambda^{-1}|\} \leq 40\tau \int_0^{\frac{1}{\tau}} ra(r)dr + d\tau^{-1} + \tau\lambda^{-1}, & (t,\tau > 0) \\ \tau^2|\Delta_\tau(t,\tau)| \leq 500 \int_0^t b(r)dr + 6000\tau^2 \int_0^{\frac{1}{\tau}} r^2 c(r)dr + 2d\tau^{-1} & (t,\tau > 0) \end{cases}$$

and from (4.28) and (4.3) we have $|D(\tau)| \geq |\mathrm{Im}\ D(\tau)| \geq d\tau^{-1}$. When $d = 0$ we have a lower bound on $|D(\tau)|$ from (4.1). Thus, referring to (6.1), we obtain

$$|\lambda^3 t^2 \mu_{41}(t,\lambda)| \leq M + M \int_0^\rho \left\{ \frac{\tau \int_0^{\frac{1}{\tau}} ra(r)dr + \tau + d\tau^{-1} + \int_0^t b(r)dr + \tau^2 \int_0^{\frac{1}{\tau}} r^2 c(r)dr}{\max\{d\tau^{-1}, \int_0^{\frac{1}{\tau}} a(r)dr\} \cdot \gamma^4} \right.$$

$$\left. + \left(\frac{\tau \int_0^{\frac{1}{\tau}} ra(r)dr + \tau + d\tau^{-1}}{\max\{d\tau^{-1}, \int_0^{\frac{1}{\tau}} a(r)dr\}} \right)^2 \gamma^{-4} \right\} d\tau$$

$\leq M \left(\int_0^t b(r)dr + 1\right)$; in other words

$$(6.4)\qquad |\mu_{41}(t,\lambda)| \leq M\, q(t) \ .$$

From (5.4) it is clear that

$$|\lambda^3 t \mu_{42}(t,\lambda)| \leq M \int_0^\rho \frac{[b(t) - tb'(t)]}{\gamma^5} d\tau \leq \frac{M\rho}{\gamma^5}(b(t) - tb'(t)) \leq \frac{M\rho}{\gamma^5} tq(t) \ .$$

Taken together with (6.4), this shows that

$$(6.5)\qquad |u_4(t,\lambda)| \leq Mq(t)$$

which is the case $j = 4$ of (4.37).

In order to obtain a similar estimate on $u_5(t,\lambda)$ we partition the interval $[\frac{\rho}{2},\infty)$ into four sets

$$[\frac{\rho}{2},\infty) = [\frac{\rho}{2},\frac{\omega}{2}) \cup [\frac{\omega}{2},\omega-\frac{\rho}{2}) \cup [\omega-\frac{\rho}{2},\omega+\frac{\rho}{2}) \cup [\omega+\frac{\rho}{2},\infty)$$

$$= E_1 \cup E_2 \cup E_3 \cup E_4 .$$

We use the estimates of Lemma 5.2 on $E_1 \cup E_2 \cup E_4$ and (2.3) on E_3 for lower bounds on $|D(\tau,\lambda)|$. Lemmas 4.1 and 5.1 will again give upper bounds on the numerators.

We know from (6.3) that

$$\text{(6.6)} \qquad |\Delta_\tau(t,\tau)| \le M(\tau^{-2}\int_0^t b(r)dr + \int_0^{\frac{1}{\tau}} r^2 c(r)dr + d\tau^{-3})$$

$$\le M_1 t^2 q(t)\tau^{-2} \qquad (\tau > \frac{\rho}{2}) ,$$

and

$$\text{(6.7)} \qquad \max\{|\Delta(t,\tau) + i\lambda^{-1}|, |D_\tau(\tau,\lambda)|\} \le M\int_0^{\frac{1}{\tau}} ra(r)dr \qquad (\tau \ge \frac{\rho}{2}) .$$

Using the estimate $\int_{\frac{1}{\omega}}^{\frac{2}{\omega}} ra(r)dr \le \frac{3}{2}\omega^{-2}a(\omega^{-1}) \le 3\int_0^{\frac{1}{\omega}} ra(r)dr$, along with (4.27) and (6.7), we obtain

$$\text{(6.8)} \qquad \max\{|\Delta(t,\tau) + i\lambda^{-1}|, |D_\tau(\tau,\lambda)|\} \le M\int_0^{\frac{2}{\omega}} ra(r)dr \le 20M\lambda^{-1} \qquad (\tau \ge \frac{\omega}{2}) .$$

Returning to (6.2), we observe that

$$\text{(6.9)} \qquad |t^2 u_{51}(t,\lambda)| \le M + \int_\rho^\infty \{\frac{|\Delta_\tau(t,\tau)|}{\lambda|D(\tau,\lambda)|^2} + \frac{2|\Delta(t,\tau) + i\lambda^{-1}||D_\tau(\tau,\lambda)|}{\lambda|D(\tau,\lambda)|^3}\}d\tau .$$

From (6.6), (6.7), (5.8), and (4.27) it follows that

(6.10)
$$\int_{E_1}\left\{\frac{|\Delta_\tau(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}+\frac{2|\Delta(t,\tau)+i\lambda^{-1}||D_\tau(\tau,\lambda)|}{\lambda|D(\tau,\lambda)|^3}\right\}d\tau$$

$$\leq Mt^2q(t)\left[\frac{1}{\lambda\int_0^{\frac{2}{\omega}} ra(r)dr}\cdot\int_{\frac{\rho}{2}}^{\frac{\omega}{2}}\frac{d\tau}{\tau^4\int_0^{\frac{1}{\tau}} ra(r)dr}+\frac{1}{\lambda\int_0^{\frac{2}{\omega}} ra(r)dr}\int_{\frac{\rho}{2}}^{\frac{\omega}{2}}\frac{d\tau}{\tau^3}\right]$$

$$\leq M_1t^2q(t)\int_{\frac{\rho}{2}}^{\infty}[\frac{1}{\tau^2}+\frac{1}{\tau^3}]d\tau\leq M_2t^2q(t)\ .$$

From (6.6), (6.8), (5.9), (4.24), and (4.27) we find that

(6.11)
$$\int_{E_2\cup E_4}\left\{\frac{|\Delta_\tau(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}+\frac{2|\Delta(t,\tau)+i\lambda^{-1}|\cdot|D_\tau(\tau,\lambda)|}{\lambda|D(\tau,\lambda)|^3}\right\}d\tau$$

$$\leq Mt^2q(t)\left[\int_{\frac{\omega}{2}}^{\omega-\frac{\rho}{2}}+\int_{\omega+\frac{\rho}{2}}^{\infty}\frac{1}{(\tau-\omega)^2}+\frac{1}{(\tau-\omega)^3}d\tau\right]$$

$$\leq M_1t^2q(t)\ .$$

Then (6.6), (6.8), (4.3), (4.27) and (2.3) yield

(6.12)
$$\int_{E_3}\left\{\frac{|\Delta_\tau(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}+\frac{2|\Delta(t,\tau)+i\lambda^{-1}|\cdot|D_\tau(\tau,\lambda)|}{\lambda|D(\tau,\lambda)|^3}\right\}d\tau$$

$$\leq Mt^2q(t)\int_{\omega-\frac{\rho}{2}}^{\omega+\frac{\rho}{2}}\left\{\frac{\theta(\omega)\theta(\tau)}{\varphi^2(\tau)}+\frac{\theta^3(\omega)}{\varphi^3(\tau)}\right\}d\tau$$

$$\leq M_1t^2q(t)\ .$$

Now (6.9) through (6.12) imply that

(6.13)
$$|\mu_{51}(t,\lambda)|\leq M\,q(t)\ .$$

It remains to establish an estimate on

(6.14) $$|t\mu_{52}(t,\lambda)| \leq \int_{\rho}^{\infty} \frac{|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \quad .$$

From (5.4) and (5.8) we argue as in (6.10) to obtain

(6.15) $$\int_{E_1} \frac{|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \leq \frac{Mtq(t)}{\lambda\left(\int_0^{\omega} ra(r)dr\right)^2} \int_{\frac{\rho}{2}}^{\frac{\omega}{2}} \frac{d\tau}{\tau^2} \leq M_1 tq(t) \quad .$$

As in (6.11), (5.4) and (5.9) yield

(6.16) $$\int_{E_2 \cup E_4} \frac{|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \leq Mtq(t) \left[\int_{\frac{\omega}{2}}^{\omega-\frac{\rho}{2}} + \int_{\omega+\frac{\rho}{2}}^{\infty} \right] \frac{d\tau}{(\tau-\omega)^2} \leq M_1 tq(t) \quad .$$

Finally, (5.4) and (2.3) give us

(6.17) $$\int_{E_3} \frac{|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \leq Mtq(t) \int_{\omega-\frac{\rho}{2}}^{\omega+\frac{\rho}{2}} \frac{\theta(\tau)\theta(\omega)}{\varphi^2(\tau)}\, d\tau \leq M_1 tq(t) \quad .$$

Combining (6.13) through (6.17) we have now established that

$$|u_5(t,\lambda)| \leq Mq(t) \quad .$$

This completes the case $j = 5$ of (4.37).

In order to prove (4.38) one need only apply the estimates of Lemmas 4.1 and 5.2 along with (4.27) and (4.33) to the functions as defined in (4.35) in the same manner as we have done in this section, noting that in this case the decomposition $a(t) = b(t) + c(t)$ from (2.1) is never used.

7. Proof of Corollary 2.1 and an example. In view of Theorem 2.2 it suffices to show that either hypothesis (i) or hypothesis (ii) of the corollary implies (2.3).

If $c \equiv 0$ and $a(0+) = b(0+) < \infty$ and $a(t)$ is strongly positive then there exist constants $\eta > 0$ and $\mu > 0$ such that

$$\varphi(\tau) \geq \frac{\eta}{1+\tau^2} \geq \mu \frac{a(0)}{\tau^2} \geq \mu \int_0^{\frac{1}{\tau}} t a(t)dt \geq \frac{\mu}{12} \theta(\tau) > 0 \qquad (\tau \geq \rho) ,$$

and

$$\varphi(\tau) \geq \frac{\eta}{1+\rho^2} > 0 \qquad (0 < \tau \leq \rho) .$$

Here we have used (4.3). This establishes (2.3) when (i) holds.

Assume now that hypothesis (ii) holds. If $a(0+) < \infty$ then [18, Corollaries 2.1 and 2.2] imply that c is strongly positive, and hypothesis (ii) assures that, for sufficiently small $x_0 > 0$, there exists $\beta > 0$ such that

$$\frac{\int_0^x a(t)dt}{\int_0^x c(t)dt} \leq \beta \qquad (0 < x \leq x_0) ,$$

and that there exists $\nu > 0$ such that

$$\mathrm{Re}\ \hat{c}(\tau) \geq \nu \tau^{-2} \qquad (\tau \geq \rho) .$$

Therefore

$$\varphi(\tau) \geq \mathrm{Re}\ \hat{c}(\tau) \geq \nu\tau^{-2} \geq \frac{\nu}{c(0)\tau} \int_0^{\frac{1}{\tau}} c(t)dt \geq \frac{\nu}{\beta\, c(0)\tau} \int_0^{\frac{1}{\tau}} a(t)dt$$

$$\geq \frac{\nu}{4\beta c(0)} \frac{|\hat{a}(\tau)|}{\tau} \geq M\ \theta(\tau) > 0 \qquad (\tau \geq \max\{\rho, x_0^{-1}\}) .$$

Here we have used (4.1). The condition (2.3i) is satisfied, because a is strongly positive. This establishes (2.3) when (ii) holds and $a(0+) < \infty$.

Assume now that (ii) holds and $a(0+) = \infty$. Then $c(0+) = \infty$, and we define

$$c_1(t) = \begin{cases} \frac{1}{2}(t-t_1)^2\, c''(t_1^-) + (t-t_1)\, c'(t_1) + c(t_1) & (0 < t \le t_1) \\ c(t) & (t > t_1) \end{cases}$$

and $c_2(t) = c(t) - c_1(t), \quad (t > 0)$.

Then c_1 and c_2 both satisfy (H), $-c_2'(t)$ is convex $c_2 \in L^1(\mathbb{R}^+)$, $c_1(0+) < \infty$. and hence $c_2(0+) = \infty$.

By a result of O. J. Staffans [23, Theorem 2(iii)]

$$\alpha = \inf_{\tau>0} \left\{ \frac{\operatorname{Re} \hat{c}_2(\tau)}{|\hat{c}_2(\tau)|^2} \right\} > 0 .$$

Furthermore $c_1(0+) < \infty$ and hypothesis (ii) imply that for some $x_0 > 0$, $\beta > 0$,

$$\frac{\int_0^x a(t)dt}{\int_0^x c_2(t)dt} < \beta < \infty \qquad (0 < x \le x_0) .$$

Thus

$$\varphi(\tau) \ge \operatorname{Re} \hat{c}_2(\tau) \ge \alpha |\hat{c}_2(\tau)|^2 \ge \frac{\alpha}{8} \left(\int_0^{\frac{1}{\tau}} c_2(t)dt \right)^2$$

$$\ge \frac{\alpha}{8\beta^2} \left(\int_0^{\frac{1}{\tau}} a(t)dt \right)^2 \ge \frac{\alpha}{8\beta^2} \frac{a(t_1)}{\tau} \int_0^{\frac{1}{\tau}} a(t)dt$$

$$\ge \frac{\alpha a(t_1)}{8\beta^2} \frac{|\hat{a}(\tau)|}{\tau} \ge M\, \theta(\tau) > 0 \qquad (\tau \ge \max\{\rho, x_0^{-1}\}) .$$

Again (2.3i) is trivial, and we have established (2.3) when $a(0+) = \infty$. This completes the proof of Corollary 2.1.

We conclude with an example where (2.3i) holds and the kernel is even strongly positive but (2.3ii) is not satisfied.

Let $b_k(t) = (1 - 2^{2^k} t)\chi_{[0,2^{-2^k}]}(t)$ $(t \geq 0,\ k = 0, 1, 2, 3, \ldots)$ where χ_E denotes the characteristic function of the set E. Define

$$a(t) = \sum_{k=0}^{\infty} b_k(t) \qquad (t > 0) .$$

This sum is finite for each $t > 0$ and one easily checks that $a(t)$ satisfies (1.3) with $a(0+) = \infty$. A direct computation shows that $a \in L^1(0,\infty)$, and

$$\varphi(\tau) = \sum_{k=0}^{\infty} 2^{2^k} \frac{(1 - \cos 2^{-2^k}\tau)}{\tau^2} \qquad (\tau > 0) . \tag{7.1}$$

We first show that $a(t)$ is strongly positive. Note that

$$\frac{1}{2} u^2 \geq 1 - \cos u \geq \frac{1}{4} u^2 \qquad (0 \leq u \leq 1) , \tag{7.2}$$

and let $m = [\log_2 \log_2 \tau]$ for $\tau \geq 2$, where $[\ \ ]$ denotes the greatest integer function. Thus $2^{2^m} \leq \tau \leq 2^{2^{m+1}}$. (7.1) and (7.2) imply that

$$\varphi(\tau) \geq \frac{1}{4} \sum_{k=m+1}^{\infty} 2^{-2^k} \geq \frac{1}{4} 2^{-2^{m+1}} = \frac{1}{4}\left[\frac{1}{2^{2^m}}\right]^2 \geq \frac{1}{4\tau^2} \qquad (\tau \geq 2) . \tag{7.3}$$

For $0 < \tau \leq 2$ we use (4.7) and (7.2) to obtain

$$\varphi(\tau) = \tau^{-2} \int_0^{\infty} (1 - \cos\tau t)\, da'(t) \geq \frac{1}{4} \int_0^{\frac{1}{\tau}} t^2\, da'(t) \geq K > 0 , \tag{7.4}$$

where $K = \frac{1}{4} \int_0^{\frac{1}{2}} t^2\, da'(t)$ is a fixed positive constant.

(7.3) and (7.4) show that $a(t)$ is strongly positive, even though $da'(t)$ is a purely singular measure. (Compare [18, Section 4].)

We next show that $a(t)$ does not satisfy (2.3ii). Let $\tau_n = 2^{2^n}(2\pi)$ $(n = 0,1,2,3,\ldots)$.

Then, referring to (7.1), (7.2) and (7.3), and using the fact that $\operatorname{Re} \hat{b}_k(\tau_n) = 0$ $(k = 0,1,2,\ldots,n)$, we find that

$$\varphi(\tau_n) \leq \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{2^{2^k} (2^{-2^k}\tau_n)^2}{\tau_n^2} = \frac{1}{2} \sum_{k=n+1}^{\infty} 2^{-2^k} \leq 2^{-2^{n+1}} = \frac{(2\pi)^2}{\tau_n^2} . \tag{7.5}$$

From (4.3) we have

$$\theta(\tau_n) \geq \frac{1}{5}\int_0^{\frac{1}{\tau}} ta(t)dt \geq \frac{1}{5}\sum_{k=1}^{n}\int_0^{\frac{1}{\tau_n}} t(1 - \tau_n t)dt = \frac{n}{30\tau_n^2} \quad . \tag{7.6}$$

Comparison of (7.5) and (7.6) shows that

$$\frac{\theta(\tau_n)}{\varphi(\tau_n)} \geq \frac{n}{120\pi^2} \to \infty \qquad (n \to \infty \quad \tau_n \to \infty) \quad ;$$

thus (2.3ii) fails to hold. We note that $da'(t)$ being a purely singular measure was not the critical aspect of this counterexample. One could use $\alpha(t) = a(t) + e^{-t}$, where $a(t)$ is as defined above, and observe the same phenomenon.

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Let $y(t,x,f)$ denote the solution of $y'(t) + \int_0^t [d + a(t-s)]L\, y(s)ds\ f(t)$, $t \geq 0$, $y(0) = x$, where $d \geq 0$ and L is a self-adjoint densely defined operator on a Hilbert space $\mathcal{H}$ with $L \geq \Lambda > 0$. Let $U(t)x = y(t,x,0)$. By analyzing a related scalar equation with parameter, we find sufficient conditions on the kernel $a(t)$ for $\|U(t)\| \to 0$ $(t \to \infty)$ and $\int_0^\infty \|U(t)\|\, dt < \infty$. These results and a resolvent formula can be combined to reveal the behavior of $y(t,x,f)$ as $t \to \infty$.

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